UNIVERSITY OF MANITOBA

DATE: April 9, 2018 FINAL EXAMINATION

TITLE PAGE

EXAMINATION: Engineering Mathematical Analysis 3 TIME: 3 hours

COURSE: MATH 3132 EXAMINER: G.I. Moghaddam

NAME: (Print in ink) ____________________

STUDENT NUMBER: ____________________

SEAT NUMBER: ____________________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 3 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 9 pages of questions and two blank pages together with a *formulas sheet*. Please check that you have all the pages. You may remove the last two pages if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 90 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	7	
2	9	
3	10	
4	11	
5	16	
6	10	
7	7	
8	20	
Total:	90	

[7] 1. Let $\mathbf{F}(x,y,z) = y^2\,\hat{\mathbf{i}} + x^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ be a vector field. Show that

$$\nabla\times(\nabla\times\mathbf{F}) - \nabla(\nabla\cdot\mathbf{F}) + \nabla^2\mathbf{F} = \mathbf{0}\,.$$

$\Big[$ In general for the vector field $\mathbf{F}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$,

$$\nabla^2\mathbf{F} = \Big(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\Big)\hat{\mathbf{i}} + \Big(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2}\Big)\hat{\mathbf{j}} + \Big(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2}\Big)\hat{\mathbf{k}}\,.\Big]$$

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[9] 2. Evaluate the surface integral $\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \cdot \hat{\mathbf{n}}\, dS$, where S is that part of the paraboloid $z = x^2 + y^2$ bounded by the surfaces $x = \pm 1$ and $y = \pm 1$ and $\hat{\mathbf{n}}$ is the unit lower normal vector.

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[10] 3. Evaluate $\oint_C [(3x^2 - 4y^3)\,dx + (4x^3 - 4xy^2)\,dy + xz\,dz]$, where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 6$ and the paraboloid $z = \sqrt{5}\,(x^2 + y^2)$, directed clockwise as viewed from the origin.

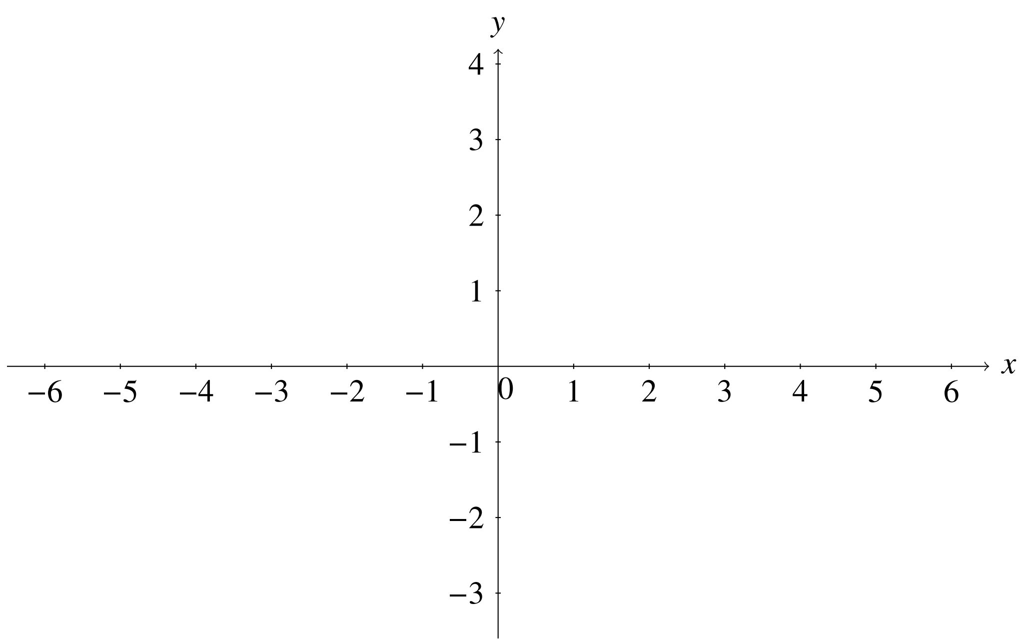
4. Consider the differential equation $2x\,y'' + (1+3x)\,y' + 3y = 0$.

[2] (a) Is $x = 0$ an ordinary point for this differential equation? Why?

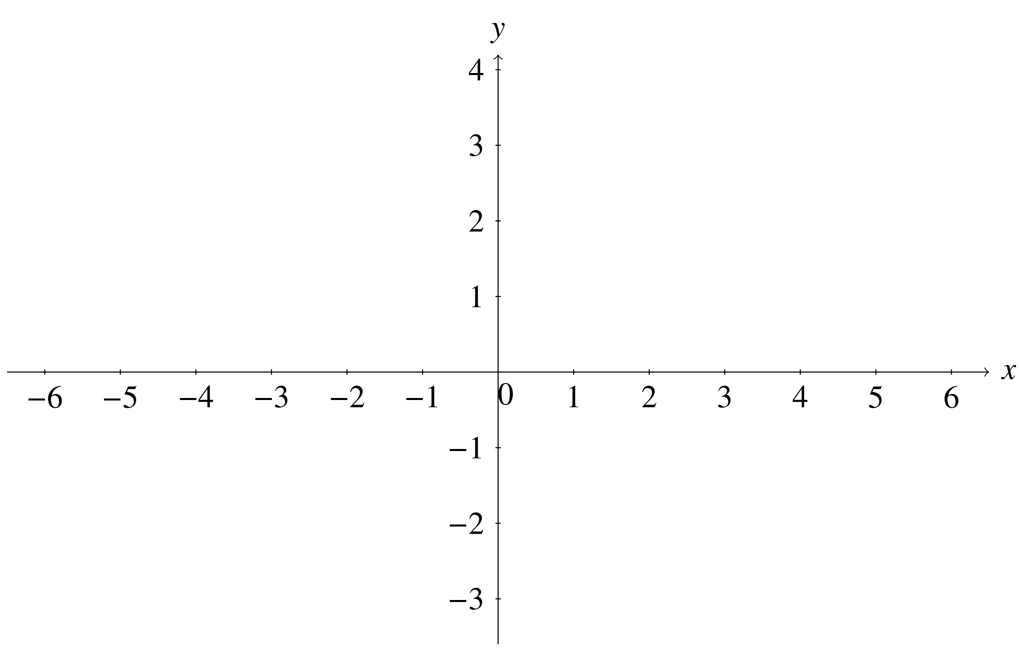
[9] (b) Use $y = \sum_{n=0}^{\infty} a_n x^n$ to solve this differential equation. Express your answer in sigma notation and simplify as much as possible. Is your solution a general solution? Why?

5. Let $f(x) = \begin{cases} 2 & \text{if } -2 < x < -1 \text{ or } 1 < x < 2 \\ x^2 & \text{if } -1 < x < 0 \text{ or } 0 < x < 1 \end{cases}$, with $f(x+4) = f(x)$.

[3] (a) Draw the graph of $f(x)$ in the interval $-6 \le x \le 6$.



[4] (b) Draw the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges, in the interval $-6 \le x \le 6$. Give an algebraic description of $g(x)$.



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[9] (c) Find the Fourier series of $f(x)$ and simplify as much as possible. Then use it to find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Hint: $\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^{k-1} & \text{if } n = 2k-1 \end{cases}$ and $\cos\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n = 2k-1 \\ (-1)^{k} & \text{if } n = 2k \end{cases}$.

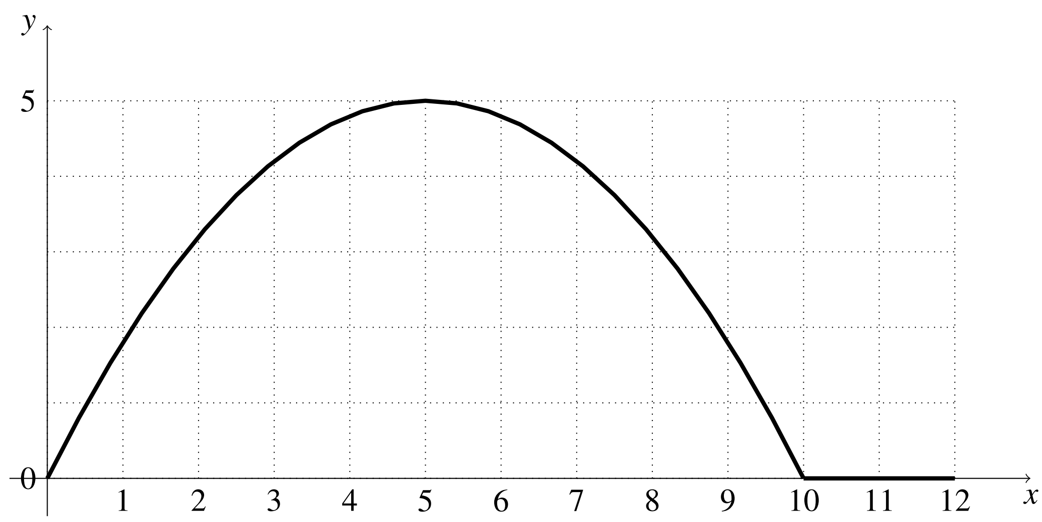
6. Consider the Sturm-Liouville system

$$\begin{aligned} y'' + 4y' + \lambda y &= 0, \qquad 0 < x < L, \\ y(0) &= 0, \\ y(L) &= 0. \end{aligned}$$

[3] (a) Find the standard form of the Sturm-Liouville system. What is the weight function?

[7] (b) Given that $\lambda \geq 4$, find all eigenvalues and eigenfunctions of this Sturm-Liouville system.

[7] 7. A string with constant linear density ρ is stretched tightly between the points $x = 0$ and $x = 12$ on the x-axis. The tension in the string is a constant τ. The displacement of the string at time $t = 0$ is shown in the figure below, and from this position, it is released. The right end of the string is fixed on the x-axis, but the left end is looped around a vertical rod, and can move vertically without friction. A restoring force proportional to displacement and also gravity are taken into account. What is the initial-value problem for displacement $y(x,t)$ of the string? Include the partial differential equation, and all boundary and initial conditions, and include intervals on which they must be satisfied.



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[20] 8. Solve the following initial boundary value problem for the temperature in a homogeneous isotropic rod with insulated sides and with no internal heat generation.

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} \quad , \quad 0 < x < L \quad , \quad t > 0 \\
U(0,t) &= 0 \quad , \quad t > 0 \\
U_x(L,t) &= 0 \quad , \quad t > 0 \\
U(x,0) &= 2L - 3x \quad , \quad 0 < x < L
\end{aligned}$$

Is there any point on the rod at which initially temperature is 0? Why?

Extra space for solution of question 8

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Sturm Liouville Systems of form $y'' + \lambda y = 0$

Boundary Conditions	Eigenvalues	Eigenfunctions
$y(0) = 0 = y(L)$	$\lambda_n = \dfrac{n^2\pi^2}{L^2}$, $n \geq 1$	$y_n(x) = \sin \dfrac{n\pi x}{L}$
$y'(0) = 0 = y'(L)$	$\lambda_n = \dfrac{n^2\pi^2}{L^2}$, $n \geq 0$	$y_0(x) = 1$, $y_n(x) = \cos \dfrac{n\pi x}{L}$
$y(0) = 0 = y'(L)$	$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n \geq 1$	$y_n(x) = \sin \dfrac{(2n-1)\pi x}{2L}$
$y'(0) = 0 = y(L)$	$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n \geq 1$	$y_n(x) = \cos \dfrac{(2n-1)\pi x}{2L}$

Answers:

Q1) $\nabla \times (\nabla \times \mathbf{F}) = (-2,-2,0)$, $\nabla(\nabla \cdot \mathbf{F}) = (0,0,2)$ and $\nabla^2\mathbf{F} = (2,2,2)$; therefore LHS= $(-2,-2,0) - (0,0,2) + (2,2,2) = (0,0,0)$ =RHS.

Q2) $\dfrac{16}{3}$

Q3) 5π

Q4) Part (a): $x = 0$ is a singular point.

Part (b): $y = a_0 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 6^n\, n!}{(2n)!} x^n$ and it is not a general solution.

Q5) Part (b): $g(x) = \begin{cases} f(x) & \text{if } x \neq n \\ 0 & \text{if } x = 4n \\ \dfrac{3}{2} & \text{if } x = 4n+1 \\ 2 & \text{if } x = 4n+2 \end{cases}$, where $n = \pm 0, \pm 1, \pm 2, \cdots$.

Part (c): $g(x) = \dfrac{7}{6} + \dfrac{2}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x - \frac{2}{\pi} \sum_{n=1}^{\infty} [(\frac{1}{2n-1} + \frac{8}{(2n-1)^3})(-1)^{n-1}] \cos \frac{(2n-1)\pi x}{2}$

then use $g(1) = \dfrac{3}{2}$ to get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Q6) Part (a): $\dfrac{d}{dx}(e^{4x}y') + \lambda(e^{4x} - 0)y = 0$ and the weight function is $p(x) = e^{4x}$.

Part (b): $y(x) = C_2 e^{-2x} \sin \dfrac{n\pi x}{L}$, $n \geq 0$.

Q7)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g - \frac{ky}{\rho}, \quad 0 < x < 12, \quad t > 0, \quad c^2 = \frac{\tau}{\rho}, \quad k > 0$$

$y(12,t) = 0\,, \quad t > 0$

$y_x(0,t) = 0\,, \quad t > 0$

$y_t(x,0) = 0\,, \quad 0 < x < 12$

$$y(x,0) = \begin{cases} 2x - \frac{1}{5}x^2 & \text{if } 0 \le x \le 10 \\ 0 & \text{if } 10 \le x \le 12 \end{cases}.$$

Q8) $U(x,t) = \displaystyle\sum_{n=1}^{\infty} \left[\frac{8L}{(2n-1)\pi} - \frac{24L\,(-1)^{n-1}}{(2n-1)^2\pi^2} \right] \sin \frac{(2n-1)\pi x}{2L}\, e^{-\frac{(2n-1)^2\pi^2}{4L^2}kt}$ and it is a formal solution.